Topic 3: Differentiation

You should recall your knowledge of differentiation from AS1.

Notation:

$$y \rightarrow \frac{dy}{dx} \rightarrow \frac{d^2y}{dx^2}$$

$$f(x) \rightarrow f'(x) \rightarrow f''(x)$$

The Chain Rule

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$$

**OR Just differentiate w.r.t. the bracket then differentiate the bracket.*

Examples

Differentiate the following:

1. $y = (2x-1)^3$

$u = 2x - 1$

$y = u^3$

$\frac{du}{dx} = 2$

$\frac{dy}{du} = 3u^2$

$\frac{dy}{dx} = 3u^2 \times 2$

$= 6u^2$

$= 6(2x-1)^2$

OR $y = (2x-1)^3$

$\frac{dy}{dx} = 3(2x-1)^2 \times 2$

$= 6(2x-1)^2$

2. Find the equation of the normal to the curve $y = \frac{5}{x^2-3}$ at the point (2,5).

$y = 5(x^2-3)^{-1}$

$\frac{dy}{dx} = -5(x^2-3)^{-2} \times 2x$

$= \frac{-10x}{(x^2-3)^2}$

when $x = 2$ $\frac{dy}{dx} = \frac{-10(2)}{(2^2-3)^2}$

$= -20$

$\therefore$ grad of normal $= \frac{1}{20}$

tangent $y = \frac{1}{20}x + c$

$(2,5)$ $5 = \frac{1}{20}(2) + c$

$5 = \frac{1}{10} + c$

$5 - \frac{1}{10} = c$

$4\frac{9}{10} = c$

normal: $y = \frac{1}{20}x + 4\frac{4}{10}$ ✓

Differentiation of e^x and $\ln x$

If $y = e^x$ then $\frac{dy}{dx} = e^x$

This is a unique function in that it is equal to its own derivative.

If $y = \ln x$ then $\frac{dy}{dx} = \frac{1}{x}$

When we differentiate $y = e^{ax}$ then $\frac{dy}{dx} = ae^{ax}$ (by the chain rule)

In more general terms:

If $y = e^{f(x)}$ then $\frac{dy}{dx} = f'(x)e^{f(x)}$

and

If $y = \ln f(x)$ then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

Examples Differentiate the following with respect to *x*.

1. $y = \ln 4x^2$ using the chain rule

$$\frac{dy}{dx} = \frac{8x}{4x^2}$$
$$= \frac{2}{x}$$

2. $y = e^{x^2}$

$$\frac{dy}{dx} = 2xe^{x^2}$$

3. $y = \ln\left(\frac{x+2}{x+3}\right)$

$$y = \ln(x+2) - \ln(x+3)$$
$$y = \frac{1}{x+2} - \frac{1}{x+3}$$

Tangents and normals to exponential and natural log functions

Example Find the equations of the tangent to the curve $y = 2e^x$ at the point where $x = 0$.

$$\frac{dy}{dx} = 2e^x$$

when $x = 0$

$$\frac{dy}{dx} = 2e^0 = 2$$

grad of tangent $= 2$

tangent is $y = 2x + c$

when $x = 0$ $y = 2e^0 = 2$

sub $(0, 2)$ into tangent

$2 = 2(0) + c$

$2 = c$

$\therefore$ tangent is

$y = 2x + 2$

Differentiation with Trigonometry

In the proofs of the differentials below, x is always given in radians.

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

The bottom four results are provided on the formula sheet.

<u>Later you should be able to prove these results using the quotient and chain rules.</u>

Using the chain rule, we can obtain these important results:

If $y = \sin f(x)$ then $\dfrac{dy}{dx} = f'(x) \cos f(x)$

If $y = \cos f(x)$ then $\dfrac{dy}{dx} = -f'(x) \sin f(x)$

Examples Find $\frac{dy}{dx}$ in each of the following cases:

1. $y = 4\sin x$

$$\frac{dy}{dx} = 4\cos x$$

2. $y = \sin(x^2)$

$$\frac{dy}{dx} = 2x\cos(x^2)$$

3. $y = (\cos x)^4$

$$\frac{dy}{dx} = 4(\cos x)^3 \times -\sin x$$
$$= -4\cos^3 x \sin x$$

4. $y = \sin^3(4x^2)$

diff wrt bracket

diff bracket

diff the angle

$$\frac{dy}{dx} = 3\sin^2(4x^2) \times \cos(4x^2) \times 8x$$
$$= 24x\sin^2(4x^2)\cos(4x^2)$$

The Product Rule

If $y = uv$ where u and v are both functions of x then,

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

The Quotient Rule

If $y = \frac{u}{v}$ where u and v are both functions of x:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

When we use the product or quotient rule it is necessary to 'tidy up' the answer at the end in order to access full marks in the examination questions. This can be a difficult skill but it is **vital** you practise it.

Example Differentiate $y = 3x^3(2x-5)^4$ w.r.t. x.

$u = 3x^3$ $\quad v = (2x-5)^4$

$\frac{du}{dx} = 9x^2$ $\quad \frac{dv}{dx} = 8(2x-5)^3$

$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$= (2x-5)^4 . 9x^2 + 3x^3 . 8(2x-5)^3$

$= 9x^2(2x-5)^4 + 24x^3(2x-5)^3$

$= 3x^2(2x-5)^3[3(2x-5)+8x]$

$= 3x^2(2x-5)^3(14x-15)$ ✓

Example Differentiate $y = \frac{(4x-3)^6}{x+2}$ w.r.t. x.

$u = (4x-3)^6$

$\frac{du}{dx} = 24(4x-3)^5$

$v = x+3$

$\frac{dv}{dx} = 1$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{1.24(4x-3)^5 - (4x-3)^6}{(x+3)^2}$

$= \frac{(4x-3)^5[24-(4x-3)]}{(x+3)^2}$

$\frac{dy}{dx} = \frac{(4x-3)^5(27-4x)}{(x+3)^2}$

Further trig differentiation

We can now prove that

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

Note: only prove them in an exam if asked to do so. You can just use the result otherwise.

Proofs

1. Let $y = \tan x = \frac{\sin x}{\cos x}$

$u = \sin x \quad v = \cos x$

$\frac{du}{dx} = \cos x \quad \frac{dv}{dx} = -\sin x$

$$\frac{dy}{dx} = \frac{v\,du/dx - u\,dv/dx}{v^2}$$

$$= \frac{\cos x(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

2. Let $y = \operatorname{cosec} x$

$y = \frac{1}{\sin x}$

$y = (\sin x)^{-1}$

$$\frac{dy}{dx} = -1(\sin x)^{-2}\cos x$$

$$= \frac{-\cos x}{\sin^2 x}$$

$$= -\left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)$$

$$= -\operatorname{cosec} x \cot x$$

***Similar method for** $y = \cot x$

***Similar method for** $y = \sec x$

Example Differentiate $y = \operatorname{cosec}(3x^2 + 5x)$ w.r.t. x.

$$\frac{dy}{dx} = -(6x+5)\operatorname{cosec}(3x^2+5x)\cot(3x^2+5x)$$

When x is given as a function of y

In some situations you can find $\frac{dy}{dx}$ by making use of the property

$$\boxed{\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}}$$

Example Find $\frac{dy}{dx}$ when $x = y^2$

$$\frac{dx}{dy} = 2y \quad \therefore \frac{dy}{dx} = \frac{1}{2y} = \frac{1}{2\sqrt{x}}$$

Implicit Differentiation

A function in which y is given clearly in terms of x is called an **EXPLICIT** function.

Functions involving x and y, where the two variables are not clearly separated as described above are called **IMPLICIT** functions.

E.g. $xy + y^2 = 2$
$3x^2 - 4xy^2 + y^3 = x^3$

Example 1 Find the equation of the normal to the curve $3x^2 - xy - 2y^2 + 12 = 0$ at the point (2,3).

$3x^2 - xy - 2y^2 + 12 = 0$

*diff each term w.r.t. x

$\frac{d(3x^2)}{dx} = 6x$

$\frac{d(xy)}{dx} = x\frac{dy}{dx} + 1.y$

by product rule

$\frac{d(2y^2)}{dx} = \frac{d(2y^2)}{dy} \times \frac{dy}{dx}$

$= 4y \times \frac{dy}{dx}$

$3x^2 - xy - 2y^2 + 12 = 0$

diff wrt x

$6x - (x\frac{dy}{dx} + 1.y) - 4y\frac{dy}{dx} = 0$

$6x - y = \frac{dy}{dx}(x + 4y)$

$\frac{dy}{dx} = \frac{6x - y}{x + 4y}$

at (2,3) $\frac{dy}{dx} = \frac{12-3}{2+12} = \frac{9}{14}$

grad normal $= -\frac{14}{9}$

normal $y = -\frac{14}{9}x + c$

at (2,3) $3 = -\frac{28}{9} + c$

$3 + \frac{28}{9} = c$

$c = 6\frac{1}{9}$

normal is $y = -\frac{14}{9}x + 6\frac{1}{9}$ ✓

Parametric Differentiation

Parametric equations are usually equations in x and y are given in terms of a third parameter t.

We can find $\frac{dy}{dt}$ and $\frac{dx}{dt}$.

From the Chain Rule $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

Finding $\frac{d^2y}{dx^2}$ using Parametric Equations

Note that

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$ from the chain rule

i.e.

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \div \frac{dx}{dt}$$

<u>Example</u> Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t, given that

$$x = \frac{1}{t}; \quad y = 3t^2 + 2$$

$x = t^{-1}$

$\frac{dx}{dt} = -t^{-2}$

$\frac{dx}{dt} = -\frac{1}{t^2}$

$\therefore \frac{dt}{dx} = -t^2$

$y = 3t^2 + 2$

$\frac{dy}{dt} = 6t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= 6t \times -t^2$

$= -6t^3$

$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dt} \times \frac{dt}{dx}$

$\frac{d\left(\frac{dy}{dx}\right)}{dt} = -18t^2$

$\frac{d^2y}{dx^2} = -18t^2 \times -t^2$

$= 18t^4$