Topic 4: Integration

Indefinite Integration

Integration is the reverse process of differentiation.

The derivatives of $y = 3x^2 + 7$, $y = 3x^2 - 5$, $y = 3x^2$ are all $\frac{dy}{dx} = 6x$.

Hence when integrating $\frac{dy}{dx} = 6x$, we can only say $y = 3x^2 + c$ where c is a constant. This is called indefinite integration, as the constant c could take any value.

We write this as

$$\int 6x dx = 3x^2 + c$$

$\int f(x)dx$ means integrate $f(x)$ wrt x.

General rule:

$$\int ax^n dx = \frac{a}{n+1}x^{n+1} + c \qquad (n \neq -1)$$

"To integrate a power of x, increase the power by 1 and divide by the new power" (the opposite process of differentiation).

Example: Integrate the following

$\int x(x-1)dx$

* get in index form first

$= \int x^2 - x \, dx$

$= \frac{x^3}{3} - \frac{x^2}{2} + c$

$\int \frac{1}{x^2} + \frac{2}{x^3} + 7 \, dx$

$= \int x^{-2} + 2x^{-3} + 7 \, dx$

$= \frac{x^{-1}}{-1} + \frac{2x^{-2}}{-2} + 7x + c$

$= -\frac{1}{x} - \frac{1}{x^2} + 7x + c$

Example: A curve has a gradient given by $\frac{dy}{dx} = 3x^2 + 5$ and passes through $(-1,4)$. Find the equation of the curve.

$y = \int 3x^2 + 5 \, dx$

$y = \frac{3x^3}{3} + 5x + c$

$y = x^3 + 5x + c$

at $(-1, 4)$

$4 = (-1)^3 + 5(-1) + c$

$4 = -1 - 5 + c$

$4 = -6 + c$

$10 = c$

$y = x^3 + 5x + 10$

Differentiate with respect to x

$$y \qquad \frac{dy}{dx} \qquad \frac{d^2y}{dx^2}$$

Integrate with respect to x

<u>Example:</u> Find y as a function of x given $y'' = 15x - 2$ and that when $x = 2$ and $y' = 25$ and $y = 20$.

Note: y'' means $\frac{d^2y}{dx^2}$ and y' means $\frac{dy}{dx}$

$$\frac{d^2y}{dx^2} = 15x - 2$$

$$\frac{dy}{dx} = \int 15x - 2\,dx$$

$$= \frac{15x^2}{2} - 2x + c$$

$$\frac{dy}{dx} = 25,\ x = 2$$

$$25 = \frac{15(2)^2}{2} - 2(2) + c$$

$$25 = 26 + c$$

$$-1 = c$$

$$\frac{dy}{dx} = \frac{15x^2}{2} - 2x - 1$$

$$y = \int \frac{15x^2}{2} - 2x - 1\,dx$$

$$y = \frac{5x^3}{2} - x^2 - x + c$$

$$y = 20,\ x = 2 \qquad 20 = \frac{5(2)^3}{2} - (2)^2 - (2) + c$$

$$20 = 14 + c$$

$$6 = c$$

$$y = \frac{5x^3}{2} - x^2 - x + 6$$

Definite Integration

The definite integral $\int_a^b \ldots dx$ is defined by:

$$\int_a^b f'(x)dx = [f(x)]_a^b = f(b) - f(a)$$

Notice the use of square brackets

The numbers a (lower limit) and b (upper limit) are called the limits of the definite integral

There is no constant of integration included (they have cancelled out)

Example: Evaluate $\int_3^4 (x-2)(x+1)\,dx$

$= \int_3^4 x^2 - x - 2\,dx$

$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_3^4$

$= \left[\frac{64}{3} - \frac{16}{2} - 8\right] - \left[\frac{27}{3} - \frac{9}{2} - 6\right]$

$= \left[\frac{16}{3}\right] - \left(-\frac{3}{2}\right)$

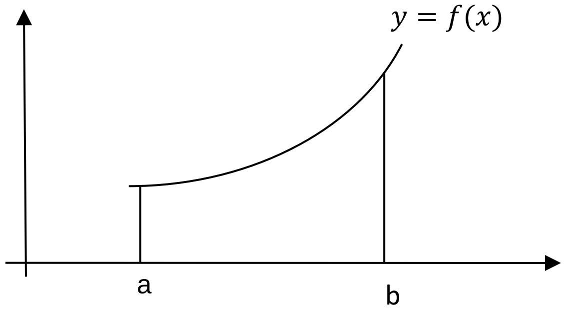
$= 6\frac{5}{6}$

Using integration to find an area

The area shown is bound by the curve $y = f(x)$, the x-axis and the lines (ordinates) $x = a$ and $x = b$.

The area is evaluated by

$$\int_a^b f(x)\,dx$$



You must always draw a sketch.

Area is positive if it is above the x-axis.

Area is negative if it is below the x-axis. But since a negative area is non sensical ignore the negative.

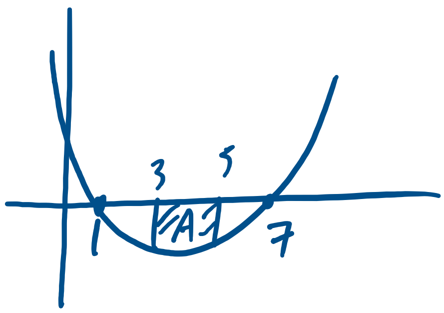
Example: Find the area bound by the curve $y = x^2 - 8x + 7$, the lines $x = 3$ and $x = 5$ and the x axis

$y = x^2 - 8x + 7$

put $y = 0$ $\quad 0 = x^2 - 8x + 7$

$0 = (x-1)(x-7)$

$x = 1, \; x = 7$



$A = \int_3^5 x^2 - 8x + 7\,dx$

$= \left[\frac{x^3}{3} - 4x^2 + 7x\right]_3^5$

$= \left[\frac{125}{3} - 100 + 35\right] - \left[\frac{27}{3} - 36 + 21\right]$

$= \left[-\frac{70}{3}\right] - \left[-6\right]$

$= -\frac{52}{3}$

$= -17\frac{1}{3}$ $\quad$ ans: $17\frac{1}{3}$ units^2